This assignment covers the whole module.

You should answer **ALL FOUR** questions. One of the requirements for passing the module is that you must obtain a score of at least 30 out of 100 for this assignment.

Question 1 – 25 marks

In this question, positions are given with reference to a Cartesian coordinate system whose x- and y-axes point due East and due North, respectively. Distances are measured in kilometres.

A light aircraft travels at a constant speed of 480 kilometres per hour (kph) in a straight line from airport A, located at position $(x_{\mathrm{A}}, y_{\mathrm{A}}) = (-200, 100)$, to airport B at $(x_{\mathrm{B}}, y_{\mathrm{B}}) = (400, -100)$.

(a) (i) Find the equation of the line of travel of the aircraft. *[3]*

(ii) Find the direction of travel of the aircraft, as a bearing, with the angle in degrees correct to one decimal place. *[2]*

(iii) Calculate the distance between airports A and B, in kilometres, correct to the nearest kilometre. *[3]*

(iv) How long does the aircraft take to travel from A to B? Give your answer in hours and minutes, correct to the nearest minute. *[3]*

(v) Find parametric equations for the line of travel of the aircraft. Your equations should be in terms of a parameter t, and should be such that the aircraft is at airport A when $t = 0$ and at airport B when $t = 1$. *[2]*

(b) During its journey, the aircraft passes a landmark, L, located at position $(90, -30)$.

(i) Let d kilometres be the distance between the location of the aircraft at parameter value t and the landmark L. Find an expression for d^2 in terms of t. Simplify your result as far as possible. *[4]*

(ii) Using your answer to part (b)(i) and the method of completing the square, determine the shortest distance, to the nearest kilometre, between the aircraft and the landmark L. *[3]*

(iii) Landmark L is visible from a distance of at most 100 kilometres. Calculate the parameter values t_1 and t_2 when the landmark L can first and last be seen from the aircraft. For how many minutes is the landmark visible from the aircraft? *[5]*

Question 2 – 25 marks

This question requires the use of Mathcad throughout, and no marks will be awarded to answers obtained by other means. For each of parts (a)–(c) you should provide an appropriate printout, though a printout on one page may cover your answers to several parts. Annotate your printouts with Mathcad text or handwriting, or reference them from a separate sheet, in order to explain clearly what you have done and what your conclusions are.

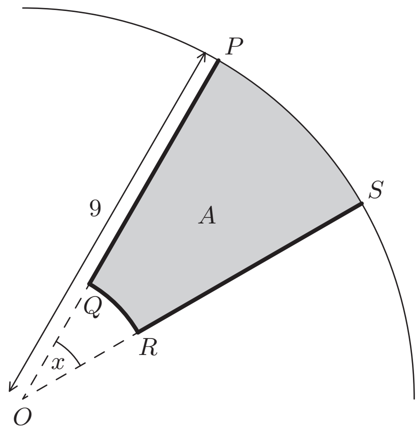
You may find it useful in all parts of this question to refer to A Guide to Mathcad.

A flexible wire $PQRS$, of total length 16 metres, is bent into a three-edged planar shape, and its ends P, S placed at the edge of a disc of radius 9 metres with centre O, as shown in the diagram below. (The arc PS is not part of the wire.) The end-segments PQ and RS of the wire lie along straight lines through O, while the arc QR forms part of a circle with centre O and subtends an angle x (in radians) at O.

This question concerns the area A enclosed between the wire and the edge of the disc, which is shown shaded below. This area can be expressed by $A = f(x)$, where

$$f(x) = \frac{x(16-9x)(20-9x)}{2(2-x)^2} \qquad \left(0 \le x \le \tfrac{16}{9}\right).$$

(*You are NOT asked to show this.*)



In part (a), you may either use the Mathcad graph plotter file (121A3-04) or plot the graph in a new worksheet of your own.

(a) (i) Plot the graph of the function $f(x)$. Your graph should cover the interval $[0, 1.78]$ in the x-direction and $[0, 60]$ in the y-direction. *[4]*

(ii) By using the 'Trace' facility (and also, if you wish, the 'Zoom' facility), estimate to two decimal places the coordinates of the point on this graph at which $y = f(x)$ takes its maximum value. *[2]*

(iii) On the same graph, plot the line $y = 36$. Using the 'Trace' facility, estimate to two decimal places both solutions of the equation $f(x) = 36$. (These solutions give the values of x for which the shaded area is $36\,\mathrm{m}^2$.) *[3]*

(b) Each of the following recurrence relations has the property that if a sequence generated by the recurrence relation converges to a limit in the interval $\left[0, \frac{16}{9}\right]$, then that limit must be a solution of the equation $f(x) = 36$. (*You are NOT asked to show this.*)

$$\mathbf{A}: \quad x_{n+1} = \frac{72(2 - x_n)^2}{(16 - 9x_n)(20 - 9x_n)} \qquad (n = 0, 1, 2, \ldots)$$

$$\mathbf{B}: \quad x_{n+1} = \frac{2}{9}\left(11 - \sqrt{\frac{72}{x_n} - 31}\right) \qquad (n = 0, 1, 2, \ldots)$$

(i) For each of these recurrence relations, generate the sequence with starting value $x_0 = 1.7$, and tabulate your results to six decimal places. Which sequence converges more rapidly? (That is, which sequence gives an estimate with specified accuracy for its limit with the smaller value of n?) [5]

(ii) Use your tabulated results to write down the solutions of the equation $f(x) = 36$ to six decimal places. [3]

(c) This part of the question concerns finding the maximum value of the function $f(x)$, as estimated in part (a)(ii), and hence the maximum possible value of the shaded area A.

You may need to put $x := x$ in your worksheet before answering part (c)(i).

(i) Use symbolic differentiation and the 'simplify' keyword to find an expression for the derivative $f'(x)$. [3]

(ii) The maximum value of $f(x)$ occurs where $f'(x) = 0$. Use a solve block to solve the equation $f'(x) = 0$ for x, giving your answer to six decimal places. [3]

(iii) Calculate the corresponding maximum possible value of the area A, giving your answer to four decimal places. [2]

Question 3 – 25 marks

As for other questions, remember to show your working explicitly throughout your answer to this question.

(a) (i) Use the Composite Rule to differentiate the function

$$f(x) = e^{(-3x+2\sin x)/6}.$$ [5]

(ii) Use the Quotient Rule and your answer to part (a)(i) to show that the function

$$g(x) = \frac{e^{(-3x+2\sin x)/6}}{3 + 2\cos x} \qquad (0 \leq x \leq 2\pi)$$

has derivative

$$g'(x) = -\frac{\left(4\sin^2 x - 12\sin x + 5\right) e^{(-3x+2\sin x)/6}}{6(3 + 2\cos x)^2}.$$ [4]

(iii) Find any stationary points of the function $g(x)$ defined in part (a)(ii), and use the First Derivative Test to classify each stationary point as a local maximum or a local minimum of $g(x)$.

(Note that the domain of the function g is the interval $[0, 2\pi]$, and recall that $|\sin x| \leq 1$ for all values of x.) [5]

(iv) Using your answers to parts (a)(ii) and (a)(iii), find the area below the graph of

$$y = \frac{20(5 - 2\sin x)(2\sin x - 1)\, e^{(-3x+2\sin x)/6}}{(3 + 2\cos x)^2} \qquad (0 \leq x \leq 2\pi)$$

and above the x-axis. Give your answer to five significant figures. [4]

(b) (i) Using your answer to part (a)(i), find the general solution of the differential equation

$$\frac{dy}{dx} = (2\cos x - 3)\, e^{(-3x+2\sin x)/6}\, y^{5/6} \qquad (y > 0),$$

giving the solution in implicit form. [4]

(ii) Find the particular solution of the differential equation in part (b)(i) for which $y = 1$ when $x = 0$, and then give this particular solution in explicit form. [3]

Question 4 – 25 marks

This question requires the use of OUStats.

(a) The file AUSTEN.OUS contains data on the lengths of 60 sentences in Jane Austen's *Sense and Sensibility.* A page was chosen at random and a sentence selected by one of two different methods: rolling a die to choose the sentence number and choosing blindly, with a pin. Thirty sentence lengths were obtained using each method of sentence selection. 'Die' contains the lengths of the sentences selected by rolling a die; 'Pin' contains the lengths of the sentences selected with a pin. [Data collected by C. E. Graham, 1996.]

(i) Use OUStats to produce boxplots of the two samples of data on a single diagram. *[2]*

(ii) Comment on what the boxplots produced in part (a)(i) tell you about the lengths of sentences obtained using each method. *[2]*

(iii) Obtain a printout of a histogram of the data for 'Pin', using the OUStats default values for the starting value of the first interval and the interval width. Add the fitted normal curve to this histogram before printing. Write down the mean and the standard deviation of the fitted normal distribution. Is this normal distribution a good fit for the data? Explain your answer. *[4]*

(iv) Use the two-sample z-test to investigate whether there is a difference between the mean length of sentences obtained by the two methods. Your answer should include the following:

- A statement of the null and alternative hypotheses, including an explanation of the meanings of any symbols that you use (other than H_0 and H_1).
- The means of the lengths of sentences obtained by the two methods, and the value of the test statistic. (You should use OUStats to find these values.)
- Your conclusion. *[8]*

(b) The file BEARS.OUS contains the results of aerial surveys carried out on each of 20 days in a particular part of Alaska. The average wind speed on each day is in 'WindSpeed', and the number of black bears sighted is in 'Bears'. The source does not report the units in which the wind speed was measured. The data are paired. [Lindgren, B.W. and Berry, D.A. (1981) *Elementary Statistics*, p. 137, Macmillan.]

(i) Obtain a printout of a scatterplot of the data with 'WindSpeed' as the explanatory variable (which must be on the x-axis). Add the least squares fit line to the plot. *[3]*

(ii) Describe any pattern that you observe in the plot. *[1]*

(iii) Obtain the equation of the least squares fit line of 'Bears' on 'WindSpeed'. If you use labels other than 'Bears' and 'WindSpeed', specify which variable represents 'Bears' and which represents 'WindSpeed'. *[2]*

(iv) Making use of the model from part (b)(iii), find the number of bears for wind speed 40. Briefly explain why this is an invalid prediction. *[3]*